G. SZEKERES

Infinite values of meromorphic matrix functions

Dedicated to N. G. de Bruijn at the occasion of his 60th birthday

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University of New South Wales, Kensington, N.S.W. Australia

1. The study of meromorphic matrix valued functions poses two fundamental questions. One is the construction of a suitable Riemann hypersurface upon which meromorphic behaviour of functions can be defined. The other is the classification of infinite values that a matrix function can take at places of meromorphic behaviour. For rational functions the two problems coincide since the infinite values taken by a rational function can be naturally identified with the ideal points adjoined to the matrix space in the construction of its Riemann hypersurface. Such a construction has been carried out by de Bruijn [1] who showed how to extend the $n \times n$ complex matrix space into a Riemann "hypersphere" so that the continuation of any rational function into the extended space becomes a continuous mapping of that space upon itself.

In view of the complexities of the construction of a Riemann hypersurface for meromorphic matrix functions it is natural to ask whether it is possible to classify infinite matrix values independently of such a construction, by following certain obvious algebraic requirements. In the present note we shall show that such a classification is indeed possible, and leads to an interesting normal form for the ideal values. The difficulties are not completely avoided; in particular, it is by no means obvious (and we shall not attempt to show) that the adjoining of the ideal values to the matrix algebra (with suitable topology) results in the same generalized Riemann sphere as the one constructed by de Bruijn. At present it is not even clear whether a rational function has a well defined ideal value at all places where it is meromorphic.

To start off with a simple example consider the value of the function X^{-1} at the 2×2 complex matrix place

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \ ad = bc, \ |a| + |b| + |c| + |d| > 0.$$

Following an earlier suggestion by K. F. Gibson and the author [2] we apply a holomorphic perturbation

$$A(t) = \begin{pmatrix} a + a_1 t + \dots, & b + b_1 t + \dots \\ c + c_1 t + \dots, & d + d_1 t + \dots \end{pmatrix}$$

where t is a complex parameter and for simplicity it is assumed that $\Delta = ad_1 + da_1 - bc_1 - cb_1 \neq 0$. Then $(A(t))^{-1} = B_0 t^{-1} + B_1 + B_2 t + \dots$ for appropriate matrices $B_0 = \frac{1}{\Delta} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$, $B_1, B_2, \dots$, and we want to associate with the principal parts $B_0 t^{-1} + B_1$ an infinite value at $t = 0$.

More generally let $\mathscr{B} = \mathscr{B}_n$ denote the Banach algebra of complex $n \times n$ matrices under the usual maximum norm, $\mathscr{P} = \mathscr{P}_n$ the family of $\mathscr{B}$-valued polynomials in t^{-1},

$$(1) \qquad A(t) = \sum_{m=0}^{k} A_m t^{m-k}, \ A_m \in \mathscr{B}, \ m = 0, 1, \dots, k, \ A_0 \neq 0, \ k > 0,$$

$\mathscr{O} = \mathscr{O}_n$ the family of $\mathscr{B}$-valued polynomials

$$X(t) = \sum_{m=0}^{r} X_m t^m, \ X_0 = I, \ X_m \in \mathscr{B}, \ m = 1, \dots, r$$

(I the identity matrix). Not all principal parts of the form (1) represent different values at $t = 0$ and we want to set up suitable criteria to decide when do two principal parts $A(t), B(t) \in \mathscr{P}$ have the same value at $t = 0$.

First, it is clear that if we want to attach a definite value of the function at a place of meromorphic behaviour, the value of $A(0)$ must be independent of the local parameter t. That is, if $\phi(\tau)$, $\psi(\tau)$ are holomorphic at $\tau = 0$, $\phi(0) = \psi(0) = 0$, and if

$$(2) \qquad A(\phi(\tau)) = B(\psi(\tau))$$

then $A(t)$, $B(t)$ must represent the same value at $t = 0$. In the case of $\mathscr{B} = \mathscr{B}_1 = \mathbb{C}$ this requirement already suffices to conclude that all principal parts $A(t) \in \mathscr{P}_1$ represent the same value at $t = 0$, that is the complex plane $\mathbb{C}$ has just one point at infinity.

For matrix algebras with $n > 1$ a more effective equivalence is needed to establish the identity of values $A(0)$ and $B(0)$. The principle that we are going to use is that if we multiply $A(t)$ from the left or from the right by a polynomial $X(t) \in \mathscr{O}$, this operation will not change the value of $A(0)$. Indeed if $f(t)$ is complex valued and holomorphic at $t = 0$ and $g(t) = f(t)h(t)$

where $h(t)$ is holomorphic with $h(0)=1$ then $g(0)=f(0)$. Moreover $\lim_{t\to 0}(f(t)-g(t))=0$, a condition that we also wish to preserve. Consequently we define an equivalence ϱ on $\mathscr{P}$ by $A(t)\varrho B(t)$, $B(t)=\sum_{m=0}^{k} B_m t^{m-k}\in\mathscr{P}$ if there exist $X(t)$, $Y(t)\in\mathcal{O}$ such that

$$B(t)\equiv X(t)A(t)Y(t)\ (\mathrm{mod}\ t).$$

Clearly ϱ is an equivalence relation since elements of $\mathcal{O}$ modulo t^{r+1} for any given $r\geqslant 0$ form a group under multiplication. We denote by $\Omega_0=\mathscr{P}/\varrho$ the set of equivalence classes under ϱ. Disregarding for the moment changes of parametrization by (1), we identify (tentatively) the ideal elements of $\mathscr{B}_n$ with elements of Ω_0.

One way of making this assumption more meaningful is to find a canonical representative of the equivalence classes which make up such an ideal element.

THEOREM 1. *Every equivalence class of $\Omega_0=\mathscr{P}/\varrho$ has a unique representative*

$$C(t)=\sum_{m=0}^{k} C_m t^{m-k},\ C_0\neq 0$$

satisfying the condition

$$C_qC_p^*=0=C_p^*C_q\ \text{for}\ 0\leqslant p<q\leqslant k \tag{3}$$

where $C^*=(c_{ij}^*)$ is the adjoint $c_{ij}^*=\bar{c}_{ji}$ of $C=(c_{ij})$.

The role of the adjoint in the theorem is essential, and the result shows that the problem of ideal values is not likely to have a neat solution for arbitrary Banach algebras unless $\mathscr{B}$ is a *-algebra. Interpreting the matrices C_m as linear operators on an n-dimensional Hilbert space $\mathscr{H}_n$ with fixed basis, equations (3) express the fact that C_q acts essentially on $\ker C_p$ and maps into $\ker C_p^*$ for all $p<q$. Indeed, let $\xi=\xi_1+\xi_2$ be the decomposition of $\xi\in\mathscr{H}_n$ into $\xi_1\in\ker C_p$, $\xi_2\in(\ker C_p)^\perp$, then $C_q\xi=C_q\xi_1$ since $\xi_2=C_p^*\xi'$ for suitable $\xi'\in(\ker C_p^*)^\perp$ and $C_q\xi_2=C_qC_p^*\xi'=0$, by the first equation (3). Moreover $C_p^*C_q\xi=0$ by the second equation (3), hence $C_q\xi\in\ker C_p^*$. Therefore C_q is essentially a linear transformation from $\ker C_p$ to $\ker C_p^*$ for every $p<q$, and we can according to Theorem 1 characterize every element of Ω_0 uniquely by a sequence $\{C_m\}_{m=0}^{k}$ where each C_m, $m=1,\dots,k$ is a linear transformation from $\bigcap_{p<m}\ker C_p$ to $\bigcap_{p<m}\ker C_p^*$. The set of these sequences is again denoted by Ω_0.

To take account of transformations of the complex parameter t, we first agree that two sequences $\{C_m\}_{m=0}^{k}$, $\{\tilde{C}_m\}_{m=0}^{k}$ of Ω_0 define the same ideal value if

$$\tilde{C}_m=\lambda^{k-m}C_m,\ m=0,1,\dots,k \tag{4}$$

for some non-zero constant λ. This is clearly an equivalence on Ω_0. Next we identify the sequences $\{\tilde{C}_m\}_{m=0}^{dk}$, $d \geqslant 1$ and $\{C_m\}_{m=0}^{k}$ if $\tilde{C}_{md} = C_m$, $m = 0, 1, \ldots, k$, $\tilde{C}_\mu = 0$ for $d \nmid \mu$. This last condition is taken care of by allowing only such sequences $\{C_m\}$ for which the g.c.d. of the set $\{m | C_{k-m} \neq 0\}$ is 1. We denote by Ω the set of equivalence classes under (4) of sequences $\{C_m\} \in \Omega_0$ satisfying this last condition, and define the elements of Ω to be the ideal elements of $\mathscr{B}_n$. If $\{C_m\}_{m=0}^{k}$ is the defining sequence of $S = S[C_0, C_1, \ldots, C_k] \in \Omega$ then k is called the degree of S; ordinary (finite) elements of $\mathscr{B}$ have degree 0.

2. To prove Theorem 1, let 0_k be the $k \times k$ zero matrix, $0_{j,k}$ the $j \times k$ zero matrix, and I_k the $k \times k$ identity matrix. Thus $0 = 0_n = 0_{n,n}$, $I = I_n$. $A_1 \oplus A_2$ is the direct sum along the diagonal of the k-dimensional A_1 and $(n-k)$-dimensional A_2,

$$A_1 \oplus A_2 = \begin{pmatrix} 0_{n-k,k} & 0_{k,n-k} \\ A_1 & A_2 \end{pmatrix}.$$

We first show:

LEMMA 1. *Given $A \in \mathscr{B} = \mathscr{B}_n$, every $B \in \mathscr{B}$ can uniquely be expressed as*

$$B = V + W$$

where $V = XA + AY$ for suitable X, $Y \in \mathscr{B}$ and $WA^* = 0 = A^*W$.

Or, if $\mathscr{V}_A$ denotes the vectorspace $\{V = XA + AY | X, Y \in \mathscr{B}\}$, $\mathscr{W}_A$ the vectorspace $\{W \in \mathscr{B} | WA^* = 0 = A^*W\}$ then

$$\mathscr{B} = \mathscr{V}_A \oplus \mathscr{W}_A. \tag{5}$$

We first note

LEMMA 2. *Given $A \in \mathscr{B}$, of rank r, there exist unitary matrices P and Q such that $PAQ = \tilde{A} \oplus 0_{n-r}$ where $\tilde{A}$ is non-singular, of dimension r.*

For proof (which is well known) simply write $A = UH$ where U is unitary and H is positive hermitian, determine a unitary Q so that Q^*HQ is diagonal with the positive entries occupying the first r diagonal positions, and set $P = Q^*U^*$.

To prove (5), determine unitary P, Q such that $D = PAQ = \tilde{A} \oplus 0_{n-r}$, $\tilde{A}$ non-singular, according to Lemma 2. Clearly $\mathscr{V}_D$ consists of all matrices $\begin{pmatrix} V_1 & V_2 \\ V_3 & 0_{n-r} \end{pmatrix}$ and $\mathscr{W}_D$ consists of all matrices $0_k \oplus \tilde{W}$ where $\tilde{W}$ is $(n-r)$ dimensional. Hence $\mathscr{B} = \mathscr{V}_D \oplus \mathscr{W}_D$. Now

$$A^*W = 0 \text{ if and only if } Q^*(A^*W)Q = (PAQ)^*(PWQ) = 0,$$
$$WA^* = 0 \text{ if and only if } P(WA^*)P^* = (PWQ((PAQ)^* = 0,$$

hence $\mathscr{W}_D = P\mathscr{W}_AQ$. Similarly $V = XA + AY$ if and only if

$$PVQ = (PXP^*)(PAQ) + (PAQ)(Q^*YQ) \text{ hence } \mathscr{V}_D = P\mathscr{V}_AQ,$$

and

$$\mathscr{B} = P\mathscr{B}Q = (P\mathscr{V}_D Q) \oplus (P\mathscr{W}_D Q) = \mathscr{V}_A \oplus \mathscr{W}_A.$$

This proves Lemma 1. Note that if $r = n$ i.e. if A is non-singular then $\mathscr{V}_A = \mathscr{H}_n$ and $\mathscr{W}_A = \{0\}$.

It follows from Lemma 1 that we can determine $X_1^{(1)}$, $Y_1^{(1)}$ in

$$X^{(1)}(t) = I + \sum_{m=1}^{k} X_m^{(1)} t^m, \quad Y^{(1)}(t) = I + \sum_{m=1}^{k} Y_m^{(1)} t^m$$

so that $B_1^{(1)} = A_1 + X_1^{(1)} A_0 + A_0 Y_1^{(1)}$ in

$$A^{(1)}(t) = X^{(1)}(t) A(t) Y^{(1)}(t) \equiv A_0 t^{-k} + \sum_{m=1}^{k} B_m^{(1)} t^{m-k} \pmod{t}$$

satisfies

$$B_1^{(1)} A_0^* = 0 = A_0^* B_1^{(1)}.$$

This $B_1^{(1)}$ is uniquely determined.

Similarly we can determine $X_m^{(1)}$, $Y_m^{(1)}$ for $m = 2, \ldots, k$ so that

(6) $$B_m^{(1)} A_0^* = 0 = A_0^* B_m^{(1)}, \quad m = 1, \ldots, k.$$

For suppose that we have already determined $X_j^{(1)}$, $Y_j^{(1)}$ for $1 \leqslant j < m$. Clearly

$$B_m^{(1)} = A_m + P_m^{(1)} + X_m^{(1)} A_0 + A_0 Y_m^{(1)}$$

where $P_m^{(1)}$ is a sum of products of matrices A_j, $X_j^{(1)}$, $Y_j^{(1)}$, $j = 1, \ldots, m-1$. Applying Lemma 1 with $A = A_0$, $B = A_m + B_m^{(1)}$, we find that $X_m^{(1)}$, $Y_m^{(1)}$ can be determined so as to satisfy (6).

If A_0 is non-singular then every $B_m^{(1)}$ is 0 and we are finished. Suppose therefore that A_0 is of rank $r < n$. Let P, Q be fixed unitary matrices such that $PA_0Q = D_0 \oplus 0_{n-r} = D_0^{(1)}$, D_0 non-singular, according to Lemma 2. Then $D_m^{(1)} = PB_m^{(1)}Q = 0_r \oplus C_m^{(1)}$ for $m = 1, \ldots, k$, because of (6), where $C_m^{(1)}$ is of dimension $n - r$. The $C_m^{(1)}$ are of course not uniquely determined. Let

$$A^{(2)}(t) = X^{(2)}(t) A(t) Y^{(2)}(t) \equiv A_0 t^{-k} + \sum_{m=1}^{k} B_m^{(2)} t^{m-k} \pmod{t}$$

also satisfy $B_m^{(2)} A_0^* = 0 = A_0^* B_m^{(2)}$ so that $D_m^{(2)} = PB_m^{(2)}Q = 0_r \oplus C_m^{(2)}$. Then for suitable

$$X^{(3)}(t) = I + \sum_{m=0}^{k} X_m^{(3)} t^m, \quad Y^{(3)}(t) = I + \sum_{m=0}^{k} Y_m^{(3)} t^m$$

we have

(7) $$X^{(3)}(t) \Big(\sum_{m=0}^{k} D_m^{(1)} t^{m-k} \Big) Y^{(3)}(t) = \sum_{n=0}^{k} D_m^{(2)} t^{m-k}.$$

We show that

(8) $\qquad X_j^{(3)} = X_j^{(4)} \oplus X_j^{(5)},\ \ Y_j^{(3)} = Y_j^{(4)} \oplus Y_j^{(5)},\ j = 1, \ldots, k$

where $X_j^{(4)}$, $Y_j^{(4)}$ are of dimension r, $X_j^{(5)}$, $Y_j^{(5)}$ of dimension $n-r$. Furthermore

(9) $\qquad X_j^{(4)} D_0 + D_0 Y_j^{(4)} = 0_r.$

For suppose that (8) is true for $1 \leqslant j < m$. Then all products $X_i^{(3)} D_k^{(1)} Y_j^{(3)}$, $i+k+j=m$, $1 \leqslant k < m$ are of the form $0_r \oplus H$, $\dim H = n-r$, and so is therefore $X_m^{(3)} D_0^{(1)} + D_0^{(1)} Y_m^{(3)}$. But $D_0^{(1)} = D_0 \oplus 0_{n-r}$, D_0 non-singular, therefore $X_m^{(3)}$, $Y_m^{(3)}$ must be of the form (8), with $X_m^{(4)} D_0 + D_0 Y_m^{(4)} = 0_r$.

It follows from (7) and (8) that

(10) $$\sum_{m=1}^{k} C_m^{(2)} t^{m-k} \equiv \Big(I_{n-r} + \sum_{m=1}^{k} X_m^{(5)} t^m\Big)\Big(\sum_{m=1}^{k} C_m^{(1)} t^{n-k}\Big)\Big(I_{n-r} + \sum_{m=1}^{k} Y_m^{(5)} t^m\Big).$$

Assuming Theorem 1 for all $A(t)$ of degree less than k (for degree 0 it is trivially true) we conclude that $C_m^{(2)}$, $m = 1, \ldots, k$ are uniquely determined by the requirement that

(11) $\qquad C_i^{(2)} C_j^{(2)*} = 0 = C_j^{(2)*} C_i^{(2)},\ \ 1 \leqslant i < j \leqslant k.$

Setting

$$C_0 = A_0,\ \ C_m = P^*(0_r \oplus C_m^{(2)}) Q^*,\ \ m = 1, \ldots, k$$

we get from (11)

$$C_i C_j^* = 0 = C_j^* C_i,\ \ 1 \leqslant i < j \leqslant k,$$

also

$$C_m C_0^* = 0 = C_0^* C_m,\ \ m = 1, \ldots, k$$

from (6), with uniquely determined C_m. This concludes the proof of Theorem 1.

3. Theorem 1 associates with $A(t)$ a unique sequence $\{C_m\}_{m=0}^k$ satisfying the condition (3). It is useful to have an explicit expression for the associated C_m in terms of the original A_m. Again we regard the A_m as linear operators on $\mathscr{H}_n$. For any subspace $\mathscr{N}$ of $\mathscr{H}_n$, denote by $P_{\mathscr{N}}$ the orthogonal projection operator into $\mathscr{N}$. Let $A = UH$, $H = \sqrt{A^*A}$ be the unique polar decomposition of $A \in \mathscr{B}_n$ into a positive self-adjoint H and a partial isometry U with $\ker U = \ker H = \ker A$ (Halmos [3], problem 105). If $H^{\dot{-}1}$ is the inverse of H on $(\ker A)^\perp = (\ker H)^\perp = (\ker U)^\perp$ then $A^{\dot{-}1} = H^{\dot{-}1} U^*$ is uniquely determined and is an "inverse" of A in the sense that

$$AA^{\dot{-}1} = P_{U\mathscr{H}} = P_{(\ker U^*)^\perp} = P_{(\ker A^*)^\perp},\ A^{\dot{-}1}A = P_{(\ker A)^\perp}.$$

THEOREM 2. *Given* $A(t) = \sum_{m=0}^k A_m t^{m-k}$, $A_m \in \mathscr{B}_n$, *define* $A_m^{(j)}$, $0 \leqslant j \leqslant m-k$ *as follows*:

Set $A_m^{(0)}=A_m,\ 0\leqslant m\leqslant k$, *and define for* $j\geqslant 0$

$$(12)\qquad A_{j+m}^{(j+1)}=\sum_{r=1}^{m}\sum_{\substack{m_1+\ldots+m_r=m\\ m_i\geqslant 1}}(-1)^{r+1}A_{j+m_1}^{(j)}C_j^{-1}A_{j+m_2}^{(j)}C_j^{-1}\ldots C_j^{-1}A_{j+m_r}^{(j)}$$

where

$$(13)\qquad C_0=A_0^{(0)}=A_0,\ C_j=P_{(C_{j-1}\mathscr{H})^\perp}\ldots P_{(C_0\mathscr{H})^\perp}A_j^{(j)}P_{\ker C_0}\ldots P_{\ker C_{j-1}}$$

for $j>0$. *Then the operators (matrices)* C_j *in Theorem* 1 *are given by the expressions* (13).

Because of the uniqueness of the C_m in Theorem 1 it is sufficient to verify that there exist $X(t),\ Y(t)\in\mathcal{O}$ such that

$$(14)\qquad \sum_{m=0}^{k}C_m t^{m-k}\equiv X(t)A(t)Y(t)\ (\mathrm{mod}\ t)$$

for the C_m defined in (13). Set

$$X_m^{(j+1)}=-A_{j+m}^{(j+1)}C_j^{-1}$$
$$Y_m^{(j+1)}=-C_j^{-1}A_{j+m}^{(j+1)},\ j=0,\ldots,k-1,\ 1\leqslant m\leqslant k-j,$$

Then

$$(I+\sum_{m=1}^{k}X_m^{(1)}t^m)(A_0t^{-k}+A_1t^{-k+1}+\ldots+A_mt^{-k+m}+\ldots)(I+\sum_{m=1}^{k}Y_m^{(1)}t^m)$$
$$\equiv C_0t^{-k}+\sum_{m=1}^{k}B_m^{(1)}t^{-k+m}\ (\mathrm{mod}\ t)$$

where

$$B_m^{(1)}=\sum_{\substack{\mu+\nu+\varrho=m\\ \mu\geqslant 0,\nu\geqslant 0,\varrho\geqslant 0}}X_\mu^{(1)}A_\nu Y_\varrho^{(1)}$$
$$=\sum_{\substack{\mu+\nu+\varrho=m\\ \mu>0,\nu\geqslant 0,\varrho>0}}A_\mu^{(1)}C_0^{-1}A_\nu^{(0)}C_0^{-1}A_\varrho^{(1)}-\sum_{\substack{\nu+\varrho=m\\ \nu\geqslant 0,\varrho>0}}A_\nu^{(0)}C_0^{-1}A_\varrho^{(1)}$$
$$-\sum_{\substack{\mu+\nu=m\\ \mu>0,\nu\geqslant 0}}A_\mu^{(1)}C_0^{-1}A_\nu^{(0)}+A_m^{(0)}.$$

Now $C_0^{-1}C_0C_0^{-1}=C_0^{-1}$ by the definition of the dot inverse, so that

$$B_m^{(1)}=\sum_{\substack{\mu+\varrho=m\\ \mu>0,\varrho>0}}A_\mu^{(1)}C_0^{-1}A_\varrho^{(1)}+\sum_{\substack{\mu+\varrho+\nu=m\\ \mu>0,\varrho>0,\nu>0}}A_\mu^{(1)}C_0^{-1}A_\nu^{(0)}C_0^{-1}A_\varrho^{(1)}$$
$$-\sum_{\substack{\nu+\varrho=m\\ \nu>0,\varrho>0}}A_\nu^{(0)}C_0^{-1}A_\varrho^{(1)}-\sum_{\substack{\mu+\nu=m\\ \mu>0,\nu>0}}A_\mu^{(1)}C_0^{-1}A_\nu^{(0)}+A_m^{(0)}-C_0C_0^{-1}A_m^{(1)}-A_m^{(1)}C_0^{-1}C_0.$$

Here the first five sums give every term in

$$A_m^{(1)}=\sum_{r=1}^{m}\sum_{\substack{m_1+\ldots+n_r=m\\ m_i\geqslant 1}}(-1)^{r+1}A_{m_1}^{(0)}C_0^{-1}A_m^{(0)}\ldots C_0^{-1}A_{m_r}^{(0)}$$

exactly once. Indeed if we count the number of occurrences of a term with $r \geqslant 3$ we get $-(m-1)+(m-2)+1+1+0=1$, similarly the number of occurrences of a term with $r=2$ is $-1+0+1+1+0=1$.

The term $A_m^{(0)}$ of course appears only once. Thus

$$\begin{aligned} B_m^{(1)} &= A_m^{(1)} - C_0 C_0^{-1} A_m^{(1)} - A_m^{(1)} C_0^{-1} C_0 \\ &= (I - P_{C_0 \mathscr{H}}) A_m^{(1)} (I - P_{C_0^{-1} \mathscr{H}}) - P_{C_0 \mathscr{H}} A_m^{(1)} P_{C_0^{-1} \mathscr{H}} \\ &= P_{\ker C_0^{-1}} A_m^{(1)} P_{\ker C_0} - P_{C_0 \mathscr{H}} A_m^{(1)} P_{C_0^{-1} \mathscr{H}}. \end{aligned}$$

Apply a further transformation

$$\Big(I + \sum_{m=1}^{k} P_{C_0 \mathscr{H}} A_m^{(0)} C_0^{-1} t^m\Big)\Big(C_0 t^{-k} + \sum_{m=1}^{k} B_m^{(1)} t^{m-k}\Big) \equiv$$

$$C_0 t^k + \sum_{m=1}^{k} P_{\ker C^{-1}} A_m^{(1)} P_{\ker C_0}\, t^{m-k} - \sum_{m=1}^{k} P_{C_0 \mathscr{H}} A_m^{(1)} P_{C_0^{-1} \mathscr{H}}\, t^{m-k}$$

$$+ \sum_{m=1}^{k} P_{C_0 \mathscr{H}} A_m^{(0)} P_{C_0^{-1} \mathscr{H}}\, t^{m-k} - \sum_{\mu+\nu=m} P_{C_0 \mathscr{H}} A_\mu^{(0)} C_0^{-1} A_\nu^{(1)} P_{C_0^{-1} \mathscr{H}}\, t^{m-k}.$$

The last two sums together are just $\sum_{m=1}^{k} P_{C_0 \mathscr{H}} A_m^{(1)} P_{C_0^{-1} \mathscr{H}}\, t^{m-k}$ and we have brought $A(t)$ to the form

$$C_0 t^{-k} + \sum_{m=0}^{k} P_{\ker C_0^{-1}} A_m^{(1)} P_{\ker C_0}\, t^{m-k},$$

By repeating the argument with $P_{\ker C_0^{-1}} X_m^{(2)} P_{\ker C_0}$ instead of $X_m^{(1)}$, $P_{\ker C_0^{-1}} A_m^{(1)} P_{\ker C_0}$, $m=1, \ldots, k$ instead of $A_m = A_m^{(0)}$ we can bring $A(t)$ to the form

$$C_0 t^{-k} + C_1 t^{-k+1} + \sum_{m=2}^{k} P_{\ker C_1^{-1}} P_{\ker C_0^{-1}} A_m^{(2)} P_{\ker C_0} P_{\ker C_1}\, t^{m-k}.$$

After k steps (by induction) we get

$$C_0 t^{-k} + C_1 t^{-k+1} + \ldots + C_k$$

where the C_m are those in (13), with $(\ker C_m)^\perp \subset \ker C_{m-1}$,

$$C_m \mathscr{H} \subset \ker C_{m-1}^{-1} = \ker C_{m-1}^*, \ m=1, \ldots, k.$$

The significance of formula (13), apart from its intrinsic interest, is that unlike Theorem 1 which does not generalize readily to bounded operators on an infinite dimensional Hilbert space, the formula can be interpreted (with some care) in arbitrary C^*-algebras and can hopefully be used as a definition of ideal elements in C^*-algebras.

4. In conclusion we touch briefly upon the question of topology of $\mathscr{B}_n = \mathscr{B}_n \bigcup \Omega$, the space obtained by adjoining the set of ideal elements

to $\mathscr{B}_n$. For $S = S[C_0, \ldots, C_k] \in \Omega$ denote by $\mathscr{S}_\delta[C_0, \ldots, C_k]$ the set

$$\{\tilde{S} = S[\tilde{C}_0, \ldots, \tilde{C}_k] \in \Omega \mid \|\tilde{C}_M - C_m\| < \delta,\ m = 0, \ldots, k\}.$$

We may then define the δ-neighbourhood of S by

$$U(S; \delta) = \mathscr{S}_\delta \bigcup \{A(\tau) \mid A(t) = A_0 t^k + \ldots + A_k \in \tilde{S} \in \mathscr{S}_\delta,\ 0 < |\tau| < \delta \cdot \|A_0\|^{1/k}\}$$

where for $\tilde{S} \in \Omega$, $A(t) \in \tilde{S}$ means member of the equivalence class of $\tilde{S}$ under ϱ and (4). The definition extends trivially to ordinary points B (of degree 0) of $\mathscr{B}_n$, when $U(B; \delta)$ becomes just an ordinary δ-neighbourhood $\{\tilde{B} \in \mathscr{B}_n \mid \|\tilde{B} - B\| < \delta\}$ of B. The neighbourhoods $U(S; \delta)$, $U(B; \delta)$ generate a Hausdorff topology for $\overline{\mathscr{B}}_n$, but whether it is the most appropriate topology for $\overline{\mathscr{B}}$ depends ultimately on whether rational functions can be continuously extended under this topology to places of meromorphic behaviour.

REFERENCES

1. Bruijn, N. G. de – Verallgemeinerte Riemannsche Sphären. Nachr. Akad. Wiss. Göttingen Math. Phys. Kl. 2., 279–292 (1959).
2. Gibson, J. K. and G. Szekeres – Analytic Matrix Functions. Nieuw Arch. Wisk. (3) 15, 233–246 (1967).
3. Halmos, P. R. – A Hilbert Space Problem Book. Van Nostrand, Princeton N.J. (1967).